Winding-angle distribution for Brownian and self-avoiding walks

Hubert Saleur
Physics Department and Math Department, University of Southern California, Los Angeles, California 90089-0484
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We study some generic aspects of the winding-angle distribution around a point in two dimensions for Brownian and self-avoiding walks using corner transfer matrix and conformal field theory.

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I. INTRODUCTION

The topological constraints on Brownian walks are a long standing subject of interest in mathematics and polymer physics. A particularly fruitful example is the study of the winding angle θ, the continuous angle swept by the movement around a set of points (curves) in two (three) dimensions (Since all results are even in θ, in what follows we generally consider $\theta \geq 0$. Otherwise, we substitute $|\theta|$ for θ.) In a plane, the probability distribution at large time or length l for the winding angle of a Brownian walk around any point O was determined by Spitzer [1] as

$$P\left[x=\frac{2\theta}{\ln l}\right]=\frac{1}{\pi}\frac{1}{1+x^2}\ , \quad l\to\infty\ . \tag{1.1}$$

In what follows, the length of the walk is usually thought of as the number of steps in a discretized version. Because of the logarithmic dependence on l, the various possible definitions of l, all proportional to each other, are equivalent. This of course is compatible with angles being dimensionless.

The law (1.1) is substantially modified when a cutoff is introduced so that the walk cannot get arbitrarily close to O. One finds then [3]

$$P\left[x=\frac{2\theta}{\ln l}\right]=\frac{\pi}{4}\frac{1}{(\cosh\pi x/2)^2}\ , \quad l\to\infty\ . \tag{1.2}$$

This cutoff leads to different results from the one in [2], where walks would bounce on the wall of the forbidden region. The standard methods of approach to these problems are based on diffusion equations, functional integrals, or refined probability theory [4,2,5].

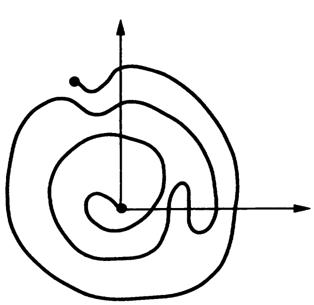

FIG. 1. A SAW on the plane winding around the origin.

More recently the same problem has been considered for self-avoiding walks (SAW's). By applying methods derived from Coulomb gas representations and conformal field theory, the equivalent of (1.1) or (1.2) (for SAW's there is no difference between the two cases as the walk provides a natural uv cutoff; see Fig. 1) was determined [6]:

$$P\left[x=\frac{\theta}{(4\ln l)^{1/2}}\right]=\frac{e^{-x^2}}{\sqrt{\pi}}\ , \quad l\to\infty\ . \tag{1.3}$$

The purpose of this paper is to study the relation between (1.2), (1.1), and (1.3) further and to compute a related distribution for SAW's that will also depend on the variable $\theta/\ln l$. The analysis uses corner transfer matrix ideas and conformal field theory. It can be considered as a follow-up of [6].

Indeed, the most striking difference between the Brownian and self-avoiding cases lies in the variable that is either $\theta/\ln l$ or $\theta/(\ln l)^{1/2}$. The intuitive explanation of this difference is that the distribution in the self-avoiding case is mainly determined by the excluded volume of the walk already wound around O (Fig. 1), while in the Brownian case, there is no such effect and the distribution is determined fully by the entropy loss that arises from the constraint of winding. To suppress this major difference and concentrate on curvature related entropy we simply put the SAW on the multisheeted Riemann surface for the function $\ln z$, discretized if necessary. For a walk of length l, we define the probability of winding angle θ by the relative number of configurations that sweep a total angle θ and for $|\theta|>2\pi$ we have end points on different sheets. It is important to realize that his problem would be *identical* to the usual winding angle problem for the Brownian case (since there is no interaction between different parts of the walk, one can just collapse the staircase onto the plane). To determine the probability distribution we can proceed in two ways.

II. CONFORMAL MAPPING

(i) First we consider a lattice SAW on a strip of length L and width W. The boundary conditions are periodic (free) in the L (W) direction (Fig. 2). On an infinite lattice the number of SAW's of length l grows as $\Omega_l \propto \mu^l$, where μ is the lattice effective coordination number. On the strip, the number of configurations for SAW's wind-

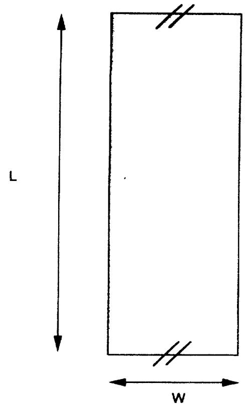

FIG. 2. Periodic strip.

ing around the periodic strip is denoted by $\Omega_l^{L,W}$. We then consider the generating function

$$\mathcal{Z}_1=\sum_l \Omega_l^{L,W}\mu^{-l} . \tag{2.1}$$

In the limit where $L/a, W/a \to \infty, L/W$ remaining finite (a is the lattice spacing), Z is determined by conformal invariance and Coulomb gas mappings arguments. It is important to notice that there is *no bulk term,* the reason being that SAW's have a fractal dimension less than 2. The rest of the procedure is standard, but we recall it very briefly for completeness. First, we choose for technical reasons the honeycomb lattice and reformulate the polymer problem as the limit $n \to 0$ of an $O(n)$ model. Partition and correlation functions of this $O(n)$ model are expressed as sums over self-avoiding mutually avoiding sets of loops and open walks, with a weight β (the inverse temperature) per monomer and n per loop [7]. For $n=0$, the critical temperature is $\beta_c=\mu^{-1}$. Next, one reformulates the $O(n)$ model in terms of the Izergin-Korepin (IK) vertex model [8,9], the correspondence between the two being conveniently carried out using quantum group symmetries [10,11]. The continuum limit of the polymer problem is then worked out using a combination of sectors of the continuum limit of the IK model, which is described by a Gaussian model with charge at infinity [12]. The net result for the lattice model is the following. The partition function where all noncontractible loops on the cylinder have weight $n=0$ has the expression

$$\mathcal{Z}_0=-\sum_{p=0}^{\infty} d_p\mathcal{K}_p+d_{p+1/2}\mathcal{K}_{p+1/2} , \tag{2.2}$$

where $\mathcal{K}_s$ is the partition function of the IK model in the spin s (integer or half integer) sector, and d_s is the quantum dimension

$$d_s=\frac{t^{2s+1}-t^{-2s-1}}{t-t^{-1}} , \tag{2.3}$$

with

$$n=-t-t^{-1} . \tag{2.4}$$

At $t=i$, using the additional level coincidences induced by $U_t sl(2)$ representation theory at q a root of unity [10], one checks that $\mathcal{Z}_0=1$ (see [13]). The computation of $\mathcal{Z}_1$ follows from the correspondence between the IK model and polymers [11,13] as

$$\mathcal{Z}_1=\frac{\partial \mathcal{Z}_0}{\partial n}-\frac{1}{2}\frac{\partial}{\partial t}\mathcal{Z}_0=\sum_{p=0}^{\infty}(p+1)(-1)^p\mathcal{K}_{p+1/2} . \tag{2.5}$$

In the continuum limit ($L/a, W/a \to \infty, L/W$ finite) we use the standard result that [13]

$$\mathcal{K}_s \to \frac{q^{h_{1+2s,1}}-q^{h_{-1-2s,1}}}{P(q)} , \tag{2.6}$$

where

$$q=\exp\left[-\frac{\pi L}{W}\right] \tag{2.7}$$

and $P(q)=\prod_{n>0}(1-q^n)$, and $h_{r,s}$ denotes the standard conformal weights in the Kac notations [14], with, in the polymer case, $h_{r,1}=[(3r-2)^2-1]/24$. We obtain, then,

$$\mathcal{Z}_1 \to Z_1=-\frac{1}{P(q)}\sum_{n=-\infty}^{\infty} n(-1)^n q^{(6n-1)(2n-1)/8} . \tag{2.8}$$

One can also consider the same quantity Z_2 defined for two SAW's, mutually avoiding, that wrap around the cylinder. One has

$$Z_2=-\frac{1}{P(q)}\sum_{n=-\infty}^{\infty} n(n+1)(-1)^n q^{3n(3n+1)/6} . \tag{2.9}$$

(ii) In the limit $L, W \to \infty$ the lattice strip can be considered as continuous and Z is a partition function for the conformal field theory that describes polymers. The various results regarding this conformal field theory can be followed through conformal mappings. We consider the mapping $z'/a=\exp(z/a)$ where z is the complex variable in the strip plane, the time direction L being along the purely imaginary axis. We then obtain the same generating function but for SAW's wrapping on a staircase (with the two end "lips" identified; see Fig. 3) of inner radius r and outer radius R in ratio $W/a=\ln(R/r)$ and angle $\theta=L/a$:

$$q=\exp\left[-\frac{\pi\theta}{\ln(R/r)}\right] . \tag{2.10}$$

To show this more precisely it is necessary to identify Z with the trace of a transfer matrix and then use known results for eigenvalues, whose properties in the mapping are deduced from the ones of correlation functions of the theory [15]. We explain this in more detail in the next section.

It is important to realize the exact meaning of the angle θ: our results make sense when r/a, $R/a \to \infty, R/r$ finite, *and* $\theta \to \infty, \theta/W$ finite, since on the strip we had $L/a \to \infty$. The above result still holds when r remains of the order of the lattice spacing, while R, θ still are very large [13,16,17]. This is due to the logarithmic dependence of W on the radius's ratio, we discuss a similar case in the next paragraph. In the following we take $r=a$.

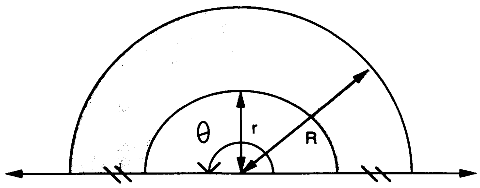

FIG. 3. By conformal mapping the strip maps onto a periodic staircase.

(iii) Third, we must discuss what becomes of this generating function when evaluated in the scaling region $\beta \approx \mu^{-1}$. (In fact, we will exclusively deal with the high temperature part $\beta < \mu^{-1}$. The behavior for low temperatures is quite different because of symmetry breaking [18].) Once again, we can rely on the $O(n)$ model analysis. Recall that the limit $\beta \to \mu^{-1}$ is the approach to the critical point, where the correlation length diverges as

$$\frac{\xi}{a} \propto (\mu^{-1} - \beta)^{-\nu} , \tag{2.11}$$

where ν is also the exponent characterizing the size of the SAW's $[\langle (R_G/a)^2 \rangle \propto l^{2\nu}$, R_G being the radius of gyration].

Now, in most problems, the knowledge of some quantity at the critical point is not enough to determine this quantity away from it. Usually the functional dependence is constrained by scaling arguments, but a nontrivial function has still to be determined. The present problem turns out to be simpler. Indeed, the usual finite size scaling hypothesis, valid in the limit $R/a, \xi/a \to \infty$, involves replacing for each of the terms in the above sums R by $Rf(\xi/R)$, where the function f depends on the term considered. One needs $f(\infty)=1$ and, also to suppress the R dependence in case the physics is dominated by the much smaller scale ξ, $f(x) \approx f_0 x$, $x \to 0$ [19]. Now suppose we consider precisely the latter regime. Since ξ still has to be very large compared to the lattice spacing and the dependence is logarithmic, f_0 disappears from (2.8) and (2.9) as $\ln(f_0 \xi/a) \approx \ln(\xi/a)$, $\xi/a \to \infty$. Therefore, we conclude that in the scaling region, (2.8) and (2.9) still hold with R/a replaced by ξ/a:

$$q = \exp\left[-\frac{\pi\theta}{\ln(\xi/a)} \right] . \tag{2.12}$$

In effect, the ir cutoff a criticality is exactly replaced by the correlation length in the scaling region.

(iv) So far we have discussed generating functions only, while the results of the most statistical interest rather deal with SAW's of large but fixed length. The standard procedure to get such results involves inverse Laplace transform [19] and is usually quite complicated for full probability distributions. Here again, the situation is simplified by the logarithmic dependence. Call Ω_l^θ the number of SAW of length l on the staircase. The scaling function F we are looking for is defined by

$$\Omega_l^\theta \approx \mu^l l^{-1} (\ln l)^\alpha F\left[\frac{\theta}{\ln l} \right] . \tag{2.13}$$

The form of (2.13) is explained as follows. For small θ we should recover the usual result for a self-avoiding loop, taking also into account the combinatorial fact that our loop can assume any position provided it encircles the origin [19]. This determines the power l^{-1} in Ω_l^θ. The exponent α is not known; it does not affect our results and we set it to zero in the following. The generating function we are computing is then

$$\mathcal{Z}_1^{\text{plane}} = \sum_l \Omega_l^\theta \beta^l \approx \int F\left[\frac{\theta}{\ln l} \right] \exp[l(\beta\mu - 1)] \frac{dl}{l} . \tag{2.14}$$

Since in the region of integration $l \gg 1$, F in (2.14) varies very slowly over the domain where the other term in the integrand takes appreciable values. This domain is around $l \propto 1/(1-\beta\mu)$. Since F depends logarithmically on l and we consider the behavior as $\beta \to \mu^{-1}$, we conclude that up to proportionality constants $\mathcal{Z}_1^{\text{plane}} \propto F[\theta/\ln(1-\beta\mu)]$, i.e., the fixed length distribution F is obtained by replacing in $\mathcal{Z}_1^{\text{plane}}$ [(2.8) or (2.9)] $\ln(\xi/a)$ by $\ln l^\nu$. In other words, the ir cutoff is now given by the average linear size of the chain. We find thus

$$F\left[\frac{\theta}{\ln l} \right] \propto -\frac{1}{P(q)} \sum_{n=-\infty}^{\infty} n(-1)^n q^{(6n-1)(2n-1)/8} , \tag{2.15}$$

where

$$q = \exp\left[-\frac{\pi\theta}{\nu \ln l} \right] . \tag{2.16}$$

The distribution function F has the following asymptotic behaviors:

$$\begin{aligned} &F(x) \propto \exp(-5\pi x/8) , \quad x \to \infty , \\ &F \propto \frac{1}{x} , \quad x \to 0 , \end{aligned} \tag{2.17}$$

where the last result follows from Poisson resummation. The divergence at small argument arises from the fact that in our computation the origin of the SAW's is not fixed, introducing an additional degree of freedom (see Sec. IV).

III. CORNER TRANSFER MATRIX

We can also derive this distribution using corner transfer matrix ideas, which are a little hidden above. We now work in the plane from the start. We can still use the polymers $O(n)$, IK model correspondence. The main difference with the computation on the strip is that the evolution is described by a corner transfer matrix (CTM) instead of a row to row one. It is important to notice that the CTM has also the property of commuting with $U_t sl(2)$, this is most clearly seen in the Hamiltonian limit where for the row to row transfer matrix $H = \sum H_i$ and for the CTM, $K = \sum i H_i$ [20], where $[H_i, U_t sl(2)] = 0$. As a result, the generating functions $\mathcal{Z}^{\text{plane}}$ have expressions similar to the ones on the cylinder, e.g.,

$$Z_1^{\text{plane}} = \sum_{p=0}^{\infty} (p+1)(-)^p \mathcal{L}_{p+1/2} , \tag{3.1}$$

where now $\mathcal{L}_s$ is the partition function of the IK model in the "radial spin s sector" [21] $\mathcal{L}=\text{Tr}(\tau_{\text{CTM}})^{\theta}$. Similar expressions appear in the computation of spontaneous magnetization for the $Q>4$ state Potts model [22]. In corner transfer matrix calculations θ is usually a multiple of $\pi/2$, but in the large angle limit we consider this becomes irrelevant.

Now the problem is to evaluate directly (without using the chain of arguments of the previous section) the continuum limit of $\mathcal{L}$ when $\beta \to \mu^{-1}, \theta \to \infty$. When $\xi/a=\infty$ it is known that the IK model corresponds to a free bosonic theory with action [7]

$$A = \int -\frac{g}{4\pi}(\partial\phi)^2 , \tag{3.2}$$

where for polymers $g=\frac{3}{2}$, ϕ is the usual free field variable, dual to the arrows of the IK model in its solid on solid reformulation [7] with normalization $\delta\phi=2\pi(\text{spin})$. Now when $\xi/a \gg 1$ this action has to be modified by adding suitable terms that make the model massive. Suppose first we were dealing with a free massive boson, i.e., $\delta A = m^2 \int \phi^2$, $m=1/\xi$. Suppose we consider a sector of radial spin s, i.e., with boundary conditions

$$\phi(r=\infty) - \phi(r=a) = -2\pi s . \tag{3.3}$$

The classical solution [i.e., satisfying $(\Delta+m^2)\phi_{cl}=0$] to this equation is

$$\phi_{cl} = 2\pi s \frac{K_0(mr)}{K_0(ma)} , \tag{3.4}$$

where K_0, K_1 are the usual Bessel functions. One has

$$(\partial\phi_{cl})^2 = 4\pi^2 s^2 m^2 \left[\frac{K_1(mr)}{K_0(ma)} \right]^2 \tag{3.5}$$

and

$$m^2\phi_{cl}^2 = 4\pi^2 s^2 m^2 \left[\frac{K_0(mr)}{K_0(ma)} \right]^2 . \tag{3.6}$$

We now have to evaluate the contributions of these two terms to the action, i.e., compute $\theta \int r\,dr$ in the limit where $ma \ll 1$. Using

$$K_0(x) \propto \ln(x) , \quad K_1(x) \propto \frac{1}{x} , \quad x \to 0 , \tag{3.7}$$

it is easy to see that the dominant contribution comes from the *kinetic term*, behaving as $1/\ln(ma)$, while the mass term gives a contribution of order $1/\ln^2(ma)$. The classical part therefore behaves as in the analysis of the critical model on the periodic strip, with the now familiar substitution $L/W \to \theta/\ln(1/ma)$. Therefore, we conclude that in the limit $ma \to 0$, $\theta \to \infty$, $\theta/\ln(ma)$ finite,

$$\mathcal{L}_s \to Z_{qu}(q^{(6s+1)^2/24} - q^{(6s+5)^2/24}) . \tag{3.8}$$

Now we must determine Z_{qu}. This is most easily done by studying the diffusion equation for Brownian motion on our Riemann surface (not surprisingly an identical problem occurs in the study of the winding angle distribution for Brownian walks [3]). Calling G the Green function one finds (recall that the region of radius a around the origin is excluded in our problem) [3]

$$G(r_<, r_>, \theta) = \int_{-\infty}^{\infty} e^{i\nu\theta} K_\nu(mr_>) \left[\frac{I_\nu(mr_<)K_\nu(ma) - I_\nu(ma)K_\nu(mr_<)}{K_\nu(ma)} \right] d\nu . \tag{3.9}$$

Using contour integration, this can be rewritten as an infinite discrete sum over the zeros (in variable ν) of the denominator, corresponding to a sum over eigenvalues of τ_{CTM}. In the limit $ma \to 0$ these zeros occur at

$$\nu_n \approx \frac{in\pi}{\ln(1/ma)} , \tag{3.10}$$

contributing to the angular dependence of the Green function by a term $\exp[\pi\theta n/\ln(ma)]$. We have therefore determined the eigenvalues of the CTM up to degeneracies. We now can rely on a mode analysis and the study of the ground state energy in [17] to conclude that

$$Z_{qu} = \frac{q^{-1/24}}{P(q)} , \tag{3.11}$$

where $q=\exp[-\pi\theta/\ln(1/ma)]$. Although the correct action describing polymers in the scaling region is rather the sine-Gordon model, it is easy to see that in the limit we are interested in this does not make any difference. All that matters is the kinetic term and the presence of a length scale $m=1/\xi$. Therefore, combining (3.8) and (3.11) we recover the results of the previous section.

The careful reader may notice that in usual corner transfer matrix computations there are additional terms, usually made of elliptic functions, that multiply the exponential of the corner transfer matrix Hamiltonian [21]. In fact, they disappear in the scaling limit we study here. Finally, computations on an integrable version of the off critical polymer problem have recently appeared, which confirm our results [23].

IV. COMMENTS

(i) The distribution (1.2) holds for a problem where the origin and end points of the Brownian walk need not be on top of each other. It looks quite difficult to obtain a similar quantity for SAW's; instead of taking a trace of the transfer matrix, one also has to insert matrix elements of boundary operators, which are known in principle but quite complicated. It is easier to derive the winding-angle distribution for Brownian loops. One finds

$$P\left[x=\frac{2\theta}{\ln l}\right]\propto\frac{1}{(\sinh\pi x/2)^2}\ , \tag{4.1}$$

which diverges at small argument like (2.17).

(ii) The distributions (4.1) and (1.2) for Brownian walks and (2.15) for SAW's are similar in nature. To see this more explicitly we expand (1.2) as

$$P\left[x=\frac{2\theta}{\ln l}\right]=\pi\exp(-\pi x)\left[\sum_{n=0}^{\infty}(-)^n\exp(-\pi n x)\right]^2$$
$$=\pi q_B\left[\sum_{n=0}^{\infty}(-)^n q_B^n\right]^2\ , \tag{4.2}$$

where

$$q_B=\exp\left[-\frac{\pi\theta}{\frac{1}{2}\ln l}\right]\ , \tag{4.3}$$

an expression similar to (2.16) since the exponent ν for Brownian walks is equal to $\frac{1}{2}$.

The uv cutoff necessary to get (1.2) and not (1.1) appears now to be very natural from the corner transfer matrix or conformal mapping point of view. Distribution (1.1) arises when this cutoff disappears and the CTM spectrum is not quantized any more. This situation is not met in most other lattice models (like the eight vertex model), where the spectrum of the CTM is discrete and in the continuum there is always an implicit uv cutoff like in the SAW case.

Let us emphasize that the integer spaced structure of the CTM spectrum *away* from the scaling region in integrable models [21] has nothing to do with the above arguments. We also do not believe that this integer spaced structure is related to quantization on angular momentum as claimed in [20]. This is especially clear in the scaling region.

(iii) The same procedure could also be applied to recover the result of [6] concerning a SAW on the plane that makes many turns, building in that fashion a core of forbidden region around O (Fig. 1). In this latter case one deals indeed with the plane, not the infinite staircase, so $L/W\to 0$ and the winding angle appears in the spin variable s. Since the dimension of the ground state of the spin s sector grows like s^2, this explains why the distribution (1.3) depends on $\theta^2/\ln l$. Let us take this opportunity to complete a point that was overlooked in [6]. The scaling form of a row to row transfer matrix eigenvalue as deduced from conformal invariance $\Lambda\approx\exp(-\pi h/W)$ holds only in the limit $W\to\infty$ with *s finite*. To establish the Gaussian distribution of [6] one needs to apply the same formula for $s\to\infty$ too, but $s\propto\sqrt{W}$ so it remains that $s\ll W$. We do not have any proof that this is legitimate, but we can give arguments based on the correction to scaling analysis [24]. Considering for instance the analytic corrections induced by the terms $L_{-2}\bar{L}_{-2}$ and $L_{-2}^2,\bar{L}_{-2}^2$ (where L_{-n} denotes the usual Virasoro algebra generators [14]), one finds that the scaled log of eigenvalue behaves as s^2+cs^4/W^2, so even as $s\propto\sqrt{W}$ the result still holds.

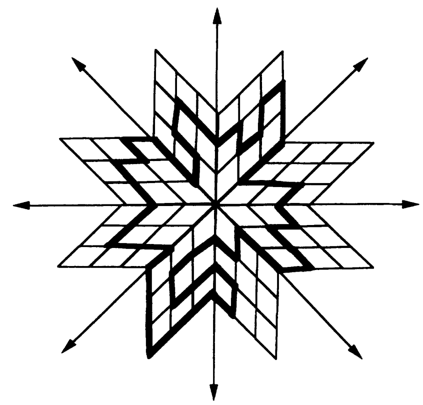

FIG. 4. A discretized staircase "unfolded" on the plane with negative curvature located at the origin and a self-avoiding loop that winds around in [with $\theta=8(\pi/2)=4\pi$].

(iv) In conclusion, the winding-angle distributions of the Brownian, SAW, and probably other geometrical problems (like percolation) around a single point can be deduced from conformal field theory only and are given by some kind of theta function. Notice that such distributions make reasonable sense as statistical problems only because the bulk free energies vanish. Otherwise the results would always be dominated by very large angles. We do not really believe that our distribution (2.15) on a staircase has much practical interest. However, it is amusing to notice that it can be reformulated slightly differently. One can "unfold" the discretized periodic staircase on the plane as in Fig. 4. Locally this is like the plane, but there is curvature at the origin. Since in our SAW problem all edges are treated as having the same length, we consider this lattice as the triangulation of a manifold with curvature located at the vertices $\rho_i=\pi(4-q_i)/q_i$, where q_i is the number of neighbors of vertex i (with the surface element $s_i=q_i/4$ this satisfies the Gauss-Bonnet theorem for closed manifolds). Distribution (2.15) appears thus as the distribution for a SAW "trapped" around a point of strong negative curvature in the limit where $l,\rho\to\infty$. Maybe in that form will it prove more useful.

ACKNOWLEDGMENTS

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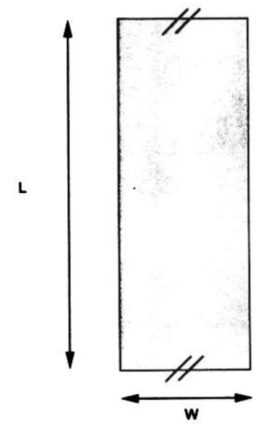

FIG. 2. Periodic strip.

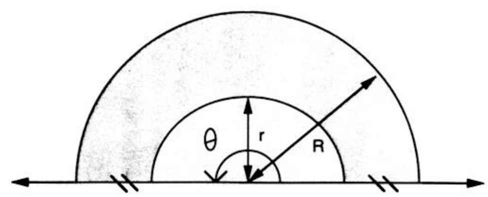

FIG. 3. By conformal mapping the strip maps onto a periodic staircase.